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# *Character of the Solutions of Certain Functional Equations.*[*]

BY THOMAS E. MASON.

### *Introduction.*

By means of a transformation of the form $z=(mx+n)/(rx+s)$, the equation[†]

$$F\left(\frac{az+b}{cz+d}\right)=\frac{A'(z)F(z)+B'(z)}{C'(z)F(z)+D'(z)}, \qquad ad-bc\neq 0,$$

in which $A'(z)$, $B'(z)$, $C'(z)$, $D'(z)$ are rational, can be transformed to the ordinary difference equation

$$\psi(x+1)=\frac{A(x)\psi(x)+B(x)}{C(x)\psi(x)+D(x)}, \tag{M}$$

or to the $q$-difference equation

$$\psi(qx)=\frac{A(x)\psi(x)+B(x)}{C(x)\psi(x)+D(x)}, \tag{N}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ are rational, according as the substitution $z'=(az+b)/(cz+d)$ has one or two double points. In order to do this it is sufficient to choose the transforming substitution so that in the first case the single double point in the $z$-plane is carried to the point infinity in the $x$-plane, and in the second case the two double points in the $z$-plane are carried to the points zero and infinity in the $x$-plane. The determination of the character of the solutions of equations (M) and (N) obviously carries with it the determination of the character of the solutions of the more general equation from which they were derived.

Tietze[‡] has investigated the solutions of equation (M) from the point of view of their transcendentally transcendental character, and Stridsberg[§] has investigated the solutions of equations (M) and (N) from the point of view of their algebraically transcendental character. Tietze normalized to the equation

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[*] Read before the American Mathematical Society, September, 1913.

[†] This equation is linear or non-linear according as $C(x)\equiv 0$ or $C(x)\not\equiv 0$.

[‡] Tietze, *Monatshefte für Mathematik und Physik*, XVI (1905), pp. 329–364.

[§] Stridsberg, *Arkiv för Matematik, Astronomi och Fysik*, VI (1910), Nos. 15 and 18. The linear equation is considered in No. 15, and the non-linear in No. 18.

$\phi(x)\{1+\phi(x+1)\}=r(x)$ and showed that if $\lim\limits_{x=\infty} r(x)=0$ and the equation has no rational solution, then it has only transcendentally transcendental solutions. Stridsberg showed that if the equations (M) and (N) have no algebraic solutions and $C(x)\not\equiv 0$, then their algebraically transcendental solutions, if any exist, also satisfy certain Riccati equations. I had proved a theorem* essentially equivalent to this important theorem of Stridsberg before the work of that investigator came to my attention.

The investigations of these men were incomplete. Neither gave a method of determining the algebraic solutions, and each left unanswered an important question—the question of whether there ever exist algebraically transcendental solutions in case the equation has no algebraic solution. Tietze says concerning the restriction of his theorem to the case when $\lim\limits_{x=\infty} r(x)=0$: "Es mag dahingestellt bleiben, ob der Satz, dass eine Gleichung (2) [$\phi(x)\{1+\phi(x+1)\}=r(x)$], die keine rationalen Lösungen hat, auch keine algebraische-transcendental Losungen besitzt, ohne Einschränkungen über $r(x)$ gilt." Stridsberg quotes these words of Tietze and follows them with the remark: "Je suis tout d' accord avec M. Tietze sur ce dernier point."

This conjecture of these men was wrong. In this paper I have demonstrated the existence of algebraically transcendental solutions in certain cases where the original equations have no algebraic solutions. The proof of existence is made to depend upon the known properties of the second-order linear difference equation.

The methods of this paper will suffice to completely characterize the solutions of any given equation of the forms (M) and (N), except where the theory concerning the solutions of the corresponding second order linear equation is incomplete.

The normal equations,

$$\phi(x)\,\phi(x+1)=R(x) \quad\text{and}\quad \phi(x)\,\phi(qx)=R(x),$$

to which certain equations of types (M) and (N) can be transformed, are of special interest in that I give simple necessary and sufficient conditions for rational and algebraic non-rational solutions; and in case there are no algebraic solutions, I prove that there are no algebraically transcendental solutions.

In § 1 the linear and non-linear equations are both normalized to type-forms, and explicit formulas are given for writing down the normal form of any given equation. It is shown that a non-linear equation can be transformed to a linear equation if a rational solution of the non-linear equation is known.

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* See § 5 of this paper.

In § 2 methods are given for determining all the rational solutions of any equation of the forms considered. The characterization of the solutions of the linear equation is completed in this section.

In § 3 algebraic solutions other than rational solutions of the normal forms of equation (N) are shown to exist in some cases, and the method of determining them is made to depend on finding the rational solutions of certain auxiliary equations.

In § 4 are given the conditions which must be satisfied in order that a function can at the same time be a solution of the functional equation (M) or (N) and of a Riccati equation. Equations are exhibited which have no algebraic solutions but which do have algebraically transcendental solutions.

In § 5 I state a condition that an equation have only transcendentally transcendental solutions. A list of several classes of equations which have only transcendentally transcendental solutions is included in this section.

## § 1. *Transformation to Normal Forms.*

Instead of writing $\chi(x+n)$ or $\chi(q^n x)$ we shall write $\chi_n(x)$ for $\chi(x+n)$ or for $\chi(q^n x)$, according as we have the ordinary difference or $q$-difference equation. We shall also write $\chi_{-n}(x)$ for $\chi(x-n)$ or for $\chi(q^{-n}x)$.

In the equation

$$\psi_1(x)=\frac{A(x)\psi(x)+B(x)}{C(x)\psi(x)+D(x)},$$

make the substitution

$$\psi(x)=\frac{\alpha(x)\Omega(x)+\beta(x)}{\gamma(x)\Omega(x)+\delta(x)}, \qquad \alpha(x)\delta(x)-\beta(x)\gamma(x)\neq 0,$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\delta(x)$ are rational. Clearing the resulting equation of fractions, we have

$$\begin{aligned}
&\Omega_1(x)\Omega(x)\{\alpha_1(x)[\alpha(x)C(x)+\gamma(x)D(x)]-\gamma_1(x)[\alpha(x)A(x)+\gamma(x)B(x)]\}\\
&\quad+\Omega_1(x)\{\alpha_1(x)[\beta(x)C(x)+\delta(x)D(x)]-\gamma_1(x)[\beta(x)A(x)+\delta(x)B(x)]\}\\
&=\Omega(x)\{\delta_1(x)[\alpha(x)A(x)+\gamma(x)B(x)]-\beta_1(x)[\alpha(x)C(x)+\gamma(x)D(x)]\}\\
&\qquad+\{\delta_1(x)[\beta(x)A(x)+\delta(x)B(x)]-\beta_1(x)[\beta(x)C(x)+\delta(x)D(x)]\}. \quad (1)
\end{aligned}$$

This equation is linear provided that the coefficient of $\Omega_1(x)\Omega(x)$ is zero or that the coefficient of $\Omega_1(x)$ and the term not containing $\Omega(x)$ are zero. If the coefficient of $\Omega_1(x)\Omega(x)$ is zero, the original equation has the rational solution $\alpha(x)/\gamma(x)$; and if the term without $\Omega(x)$ is zero, it has the rational solution $\beta(x)/\delta(x)$. If $A(x)+B(x)=C(x)+D(x)$, there is the obvious solution $\psi(x)=1$, and the equation can be transformed to a linear equation and studied in that form. Any other rational solution will effect this transformation to a linear equation. If there is no obvious rational solution, we choose $\alpha(x)=\gamma(x)=1$, $\beta(x)=B(x)-D(x)$ and $\delta(x)=C(x)-A(x)$. These values of $\alpha(x)$, $\beta(x)$, $\gamma(x)$, $\delta(x)$ reduce the coefficient of $\Omega_1(x)$ to zero, but do not make the coefficient of $\Omega_1(x)\,\Omega(x)$ or the term without $\Omega(x)$ zero, unless the original equation has the solution $\psi(x)=1$. Making these substitutions in (1) and reducing, we have

$$\Omega_1(x)=\frac{g(x)}{\Omega(x)}-h(x),$$

where

$$g(x)=\frac{(B(x)C(x)-A(x)D(x))\,(A_1(x)+B_1(x)-C_1(x)-D_1(x))}{A(x)+B(x)-C(x)-D(x)}$$

and

$$h(x)=-\frac{(A(x)+B(x))\,(A_1(x)-C_1(x))+(C(x)+D(x))\,(B_1(x)-D_1(x))}{A(x)+B(x)-C(x)-D(x)}.$$

If $h(x)\not\equiv 0$, the substitution

$$\Omega(x)=h_{-1}(x)\,\phi(x)$$

will transform the equation to

$$\phi_1(x)=\frac{R(x)}{\phi(x)}-1, \tag{2}$$

where

$$R(x)=\frac{g(x)}{h_{-1}(x)\,h(x)}.$$

If $h(x)\equiv 0$, the substitution

$$\Omega(x)=\frac{(g_{-1}(x)-g(x))\,\phi(x)-g(x)\,(1-g_{-1}(x))}{(g_{-1}(x)-g(x))\,\phi(x)-(1-g_{-1}(x))}$$

will give an equation of the form (2), where

$$R(x)=\frac{g(x)\,(1-g_1(x))\,(1-g_{-1}(x))}{(g(x)-g_{-1}(x))\,(g_1(x)-g(x))}.$$

The determinant of the transformation and the denominator of the expression for $R(x)$ can be zero only when $g(x)=g$, a constant, in which case the equation $\Omega_1(x)=g/\Omega(x)$ has the obvious rational solution $\Omega(x)=\sqrt{g}$ and can be transformed to a linear equation.

Thus, we have shown that any non-linear equation of the forms considered can be transformed to the normal form (2) or to a linear equation. It is obvious that the transformations here carried out have not affected the solutions as to their rational, algebraic non-rational, algebraically transcendental, or transcendentally transcendental character. If a linear equation is transformed to the normal form (2), it is obvious that the normal form has a rational solution; for the transformation is reversible, and a necessary and sufficient condition that a non-linear equation of this form transform to a linear equation by means of a linear fractional transformation, is that the non-linear equation have a rational solution.

The equation in $\psi(x)$ can be transformed to the special normal form

$$\phi_1(x) = \frac{R(x)}{\phi(x)} \tag{3}$$

in certain cases; that is, when the transformation can be chosen so as to make the coefficients of $\Omega_1(x)$ and $\Omega(x)$ in (1) equal to zero.

In the non-homogeneous linear equation

$$\psi_1(x) = A(x)\,\psi(x) + B(x),$$

make the substitution

$$\psi(x) = \alpha(x)\,\phi(x) + \beta(x).$$

Thus we have

$$\phi_1(x) = \frac{A(x)\,\alpha(x)}{\alpha_1(x)}\phi(x) + \frac{A(x)\,\beta(x) + B(x) - \beta_1(x)}{\alpha_1(x\ )}.$$

This becomes homogeneous if the original equation has a rational solution and $\beta(x)$ is chosen as that solution. In any event we can choose $\beta(x) = 0$ and $\alpha(x) = B_{-1}(x)$, and the equation takes the form

$$\phi_1(x) = R(x)\,\phi(x) + 1. \tag{4}$$

## § 2. *Rational Solutions of the Normal Equations and Other Solutions of the Linear Equations.*

The equation

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \tag{2}$$

has two very simple cases. If $R(x)$ is a constant, there is always a rational solution $\phi(x)$ = a constant. In the case of the $q$-difference equation, if $q$ is an $n$-th root of unity, the solutions can be obtained by eliminating from the following equations the functions $\phi$ with arguments other than $x$, and solving the resulting algebraic equation of second degree for $\phi(x)$:

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1,$$

$$\phi(q^2x) = \frac{R(qx)}{\phi(qx)} - 1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$\phi(q^{n-1}x) = \frac{R(q^{n-2}x)}{\phi(q^{n-2}x)} - 1,$$

$$\phi(x) = \frac{R(q^{n-1}x)}{\phi(q^{n-1}x)} - 1.$$

For the case of the ordinary difference equation and of the $q$-difference equation when $q$ is not a root of unity, we seek to determine an upper bound to the number of rational solutions of equation (2) by substituting a power series for $\phi(x)$. This method gives at most two determinations for the first coefficient of $\phi(x)$. Suppose that two solutions $\phi(x)$ and $\psi(x)$ have the same first coefficient; we desire to know whether they are identical. We have

$$\phi(x)\{1+\phi_1(x)\} = R(x) = \psi(x)\{1+\psi_1(x)\},$$

or

$$\frac{1+\phi_1(x)}{1+\psi_1(x)} = \frac{\psi(x)}{\phi(x)}.$$

Now suppose

$$\phi(x) = a_n x^n + \ldots\ldots + a_{n-t+1}x^{n-t+1} + a_{n-t}x^{n-t} + \ldots\ldots,$$
$$\psi(x) = a_n x^n + \ldots\ldots + a_{n-t+1}x^{n-t+1} + b_{n-t}x^{n-t} + \ldots\ldots, \qquad b_{n-t} \neq 0,$$

where $a_{n-t} \neq b_{n-t}$, but the corresponding preceding coefficients are equal.

In the ordinary difference equation the expansions for $\phi_1(x)$ and $\psi_1(x)$ when rearranged as descending power series, will have each term after the first modified by the addition of elements from the terms preceding. Since the expansions for $\phi(x)$ and $\psi(x)$ are alike up to the term containing $x^{n-t}$, the coefficients of $x^{n-t}$ in the expansions of $\phi_1(x)$ and $\psi_1(x)$ will be of the form $a_{n-t}+c$ and $b_{n-t}+c$, respectively. We shall thus have

$$\frac{\psi(x)}{\phi(x)} = 1 + \frac{b_{n-t}-a_{n-t}}{a_n}x^{-t} + \ldots\ldots, \tag{5}$$

and

$$\frac{1+\phi_1(x)}{1+\psi_1(x)} = 1 + \frac{a_{n-t}-b_{n-t}}{a_n+\delta_{0n}}x^{-t} + \ldots\ldots, \qquad \delta_{0n} = \begin{cases} 0 \text{ if } n \neq 0, \\ 1 \text{ if } n = 0, \end{cases} \tag{6}$$

provided that we do not have simultaneously $n=0$ and $a_0=-1$.

When $n=0$ and $a_0=-1$, the first term of the expansion for $1+\psi_1(x)$ is $a_{-k}x^{-k}$, $k \geqq 1$. In this event, the second quotient above is either

$$1 + \frac{a_{-t}-b_{-t}}{a_{-k}}x^{-t+k} + \ldots\ldots, \quad \text{or} \quad \frac{a_{-t}}{b_{-t}} + \ldots\ldots,$$

according as $k<t$ or $k=t$. In the first case there is no term in the quotient $\psi(x)/\phi(x)$ to compare with the term in $x^{-t+k}$, and hence $a_{-t}=b_{-t}$; in the second case $a_{-t}/b_{-t}=1$; that is, $a_{-t}=b_{-t}$.

Comparing the expansions (5) and (6), we have, for $n \neq 0$, $b_{n-t} = a_{n-t}$. Comparing for $n = 0$, we have either $a_{-t} = b_{-t}$ or $a_0 = -1/2$. When $n = 0$ and $a_0 = -1/2$, the expansion for $R(x)$ in descending powers of $x$ obviously starts with the constant $-1/4$.

Since there are at most two ways for the expansion for $\phi(x)$ to start, the work above shows that there are at most two formal power-series solutions, except when the expansion for $R(x)$ in descending powers of $x$ starts with the constant $-1/4$. The number of rational solutions is not greater than the number of power-series solutions.

We have thus demonstrated the following theorem:

*The equation*

$$\phi(x+1) = \frac{R(x)}{\phi(x)} - 1$$

*can have at most two rational solutions, except possibly when the expansion for* $R(x)$ *in descending powers of* $x$ *starts with the constant** $-1/4$.

For the $q$-difference equation we have

$$\frac{\psi(x)}{\phi(x)} = 1 + \frac{b_{n-t} - a_{n-t}}{a_n} x^{-t} + \dots .$$

and

$$\frac{1+\phi_1(x)}{1+\psi_1(x)} = 1 + \frac{a_{n-t} - b_{n-t}}{a_n + \delta_{0n}} q^{n-t} x^{-t} + \dots , \qquad \delta_{0n} = \begin{cases} 0 \text{ if } n \neq 0, \\ 1 \text{ if } n = 0, \end{cases}$$

provided that we do not have simultaneously $n = 0$ and $a_0 = -1$.

When $n = 0$ and $a_0 = -1$, the same argument as that given for the ordinary difference equation will show that we must have $a_{-t} = b_{-t}$. When this case does not arise, by comparing coefficients we readily have either

$$a_{n-t} = b_{n-t} \qquad \text{or} \qquad a_n(q^n + q^{n-t}) + \delta_{0n} = 0.$$

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* That there is an actual exception in this case is shown by the following examples: If $R(x)$ is taken equal to $-1/4$, equation (2) has the solutions

$$\phi(x) = -1/2 \qquad \text{and} \qquad \phi(x) = -\frac{x+a}{2(x+a-1)},$$

where $a$ is arbitrary. Another example can be obtained from the equation given by Tietze (*loc. cit.*, foot-note, p. 330) to show the existence of rational and algebraically transcendental solutions at the same time. The equation

$$\phi(x)\{1+\phi(x+1)\} = -\frac{(r(x) - r(x-1))\,(r(x+2) - r(x+1))}{(r(x+1) - r(x-1))\,(r(x+2) - r(x))},$$

where $r(x)$ is any rational function, has the solutions

$$\phi(x) = -\frac{r(x) - r(x-1)}{r(x+1) - r(x-1)} \qquad \text{and} \qquad \phi(x) = -\frac{(r(x) - r(x-1))\,(u(x) + r(x+1))}{(r(x+1) - r(x-1))\,(u(x) + r(x))},$$

where $u(x)$ satisfies the equation $u(x+1) = u(x)$. If $u(x)$ is chosen a constant, we have a rational solution with an arbitrary constant.

This latter equation can be satisfied only when $n \neq 0$ and $q$ is a root of unity, or when $n = 0$ and $a_0 = -q^t/(1+q^t)$. There are, therefore, for the $q$-difference-equation case at most two rational solutions, except possibly when the expansion for $R(x)$ in the neighborhood of infinity starts with the constant $-q^t/(1+q^t)^2$. If we use expansions valid in the neighborhood of the origin, we find that there are at most two rational solutions, except possibly when the expansion for $R(x)$ starts with the same constant as for the expansion at infinity determined above.

Consequently, we are led to the following theorem:

*The equation*

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1$$

*has at most two rational solutions, except possibly when the expansions for $R(x)$ in the neighborhood of the origin and of infinity start with the constant term* $-q^t/(1+q^t)^2$, where $t$ is an integer.*

The power-series expansions, while of value in determining an upper bound to the number of rational solutions, are not effective for the actual construction of these solutions. The following method, however, is practicable and suffices to determine *all* the rational solutions of any given equation of the types under consideration.

If the equation has a rational solution $\phi(x)$, we can write

$$\phi(x) = aP(x)/Q(x) \quad \text{and} \quad R(x) = r\rho(x)/\pi(x),$$

where $a$ and $r$ are constants and $P(x)$, $Q(x)$, $\rho(x)$, $\pi(x)$ are polynomials with the leading coefficients unity, and the fractions $P(x)/Q(x)$ and $\rho(x)/\pi(x)$ are in their lowest terms. We have

$$a\frac{P(x)}{Q(x)}\left(\frac{aP_1(x) + Q_1(x)}{Q_1(x)}\right) = r\frac{\rho(x)}{\pi(x)}.$$

Now, $P(x)$ and $Q_1(x)$ may have a common divisor, as may also $aP_1(x) + Q_1(x)$ and $Q(x)$. Write

$$\frac{P(x)}{Q_1(x)} = \frac{\alpha(x)}{\beta(x)} \quad \text{and} \quad a\left(\frac{aP_1(x) + Q_1(x)}{Q(x)}\right) = r\frac{\gamma(x)}{\delta(x)}, \tag{7}$$

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* An exception will be shown if we choose $R(x) = -2/9$, where $q = 2$ and $t = 1$. With this value for $R(x)$, equation (2) will have the solutions

$$\phi(x) = -1/3, \qquad \phi(x) = -2/3, \qquad \phi(x) = \frac{-2x^3 - mx^2 - m^2x + 1}{3(x^3 - 1)}, \qquad \phi(x) = \frac{-2(x+a)}{3(x+2a)},$$

where $m$ takes each of the values of the cube roots of unity, and $a$ is an arbitrary constant. Another example could be obtained if, in the example of Tietze already given, we replace the ordinary difference by the $q$-difference.

where $\alpha(x)/\beta(x)$ and $\gamma(x)/\delta(x)$ are in their lowest terms. Then $\alpha(x)\gamma(x) = \rho(x)$ and $\beta(x)\delta(x) = \pi(x)$. Combining the two equations (7) to eliminate $P(x)$, we have

$$a\,\alpha_1(x)\,\delta(x)\,Q_2(x) + a\,\beta_1(x)\,\delta(x)\,Q_1(x) - r\,\beta_1(x)\,\gamma(x)\,Q(x) = 0. \qquad (8)$$

Hence, if there is a rational $\phi(x)$, it must be possible to separate $\rho(x)$ into factors $\alpha(x)$ and $\gamma(x)$, and $\pi(x)$ into factors $\beta(x)$ and $\delta(x)$ in such a way that this equation in $Q(x)$ has a solution which is a polynomial with leading coefficient unity. This value of $Q(x)$ must also, when substituted in the first equation of (7), give $P(x)$ a polynomial with leading coefficient unity. Direct substitution in equation (8) of a polynomial $Q(x)$ of degree $n$ will determine $a$, $n$ and the coefficients of this polynomial, if any exists. Since there are only a finite number of ways of breaking $\rho(x)$ and $\pi(x)$ up into polynomial factors, a finite number of trials will give all the rational solutions of the original equation.

Turning now to the special normal form (3), we shall prove the following theorem:

*A necessary and sufficient condition that the equation*

$$\phi_1(x) = \frac{R(x)}{\phi(x)}$$

*have a rational solution, is as follows:*

1) *To every finite zero* [pole] of $R(x)$, except at the point $x = 0$ for the q-difference equation, there is a congruent zero [pole] at an odd number of steps away,† or a congruent pole [zero] at an even number of steps away.*

2) *In the case of the q-difference equation, if $x = 0$ is a zero or pole of the m-th order, then m is even.*

In the case $R(x)$ = a constant, there are no zeros or poles, and the conditions of the theorem are satisfied.

If $R(x)$ has a zero [pole] at $a$, then, from the equation, we see that either $\phi(x)$ has a zero [pole] at $a$ or $\phi_1(x)$ has a zero [pole] at that point. We shall consider the consequences of the two alternatives.

(1) $\phi(x)$ has a zero [pole] at $a$; then $\phi_1(x)$ has a zero [pole] at $a_{-1}$. If $R(x)$ has no zero [pole] at $a_{-1}$, then $\phi(x)$ must have a pole [zero] at $a_{-1}$,

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* A zero [pole] of multiplicity $m$ is to be counted as $m$ simple zeros [poles], and the zeros or poles which balance it need not be at the same point. Thus, a zero of order 5 may be balanced by a congruent zero of order 3 an odd number of steps away (see next foot-note) and a pole of order 2 at an even number of steps away.

† If there is a zero or pole at $a$, then a zero or pole at $a + n$ [$aq^n$] will be said to be congruent on the right at $n$ steps away from $a$, and a zero or pole at $a - n$ [$aq^{-n}$] will be said to be congruent on the left at $n$ steps away from $a$. The first will be denoted by $a_n$ and the second by $a_{-n}$.

in which case $\phi_1(x)$ has a pole [zero] at $a_{-2}$. Then either $R(x)$ has a pole [zero] at $a_{-2}$ or $\phi(x)$ has a zero [pole] at $a_{-2}$, and consequently $\phi_1(x)$ a zero [pole] at $a_{-3}$. Then $R(x)$ has a zero [pole] at $a_{-3}$ or $\phi(x)$ has a pole [zero] at $a_{-3}$, etc.

(2) $\phi_1(x)$ has a zero [pole] at $a$; then $\phi(x)$ has a zero [pole] at $a_1$, in which case $R(x)$ has a zero [pole] at $a_1$ or $\phi_1(x)$ has a pole [zero] at $a_1$. If $\phi_1(x)$ has a pole [zero] at $a_1$, then $\phi(x)$ has a pole [zero] at $a_2$ and $R(x)$ must either have a pole [zero] at $a_2$ or $\phi_1(x)$ a zero [pole] at that point. If $\phi_1(x)$ has a zero [pole] at $a_2$, then $\phi(x)$ has a zero [pole] at $a_3$ and $R(x)$ has a zero [pole] or $\phi_1(x)$ has a pole [zero] at $a_3$, etc.

It is thus seen that, if the zeros and poles are not arranged as stated in the theorem, the function $\phi(x)$ will have an infinite number of congruent zeros and poles. The condition is therefore necessary to the existence of a rational solution.

That the condition is sufficient is seen in the fact that, if the zeros and poles of $R(x)$ occur as stated in the theorem, we can at once write down a rational solution.

We have shown that a non-linear equation of type (2) can be transformed to a linear equation, provided that the non-linear equation has a rational solution; and we shall now consider the solutions of the linear form. Thus we shall complete the theory of the non-linear equation when it has a rational solution. In § 1 we have shown that the non-homogeneous equation can be reduced to the homogeneous equation, if a rational solution is known, and in any case it can be transformed to the normal form

$$\phi_1(x) = R(x)\,\phi(x) + 1. \tag{4}$$

We shall now show how all the rational solutions of equation (4) can be found.

If, in any set of congruent poles, $\phi(x)$ [$\phi_1(x)$] has a pole to the left [right] of the leftmost [rightmost] pole of $R(x)$, then $\phi_1(x)$ [$\phi(x)$] has a pole at one step (see foot-note, p. 000) to the left [right], and to balance the poles would require that $\phi(x)$ have an infinite number of poles. The order of any pole of $\phi(x)$ can not be greater than the sum of the orders of the congruent poles and zeros of $R(x)$. If $\phi(x)$ has a pole to which there is no congruent pole or zero of $R(x)$, it must have an infinite number. By substituting a power series in descending powers of $x$ for $\phi(x)$ and finding the degree of its first term, we can determine the relative number of zeros and poles of $\phi(x)$. Knowing the possible location of the poles and the relative number of poles and zeros, we have reduced the finding of the rational solutions of this normal form to the solving of a finite number of simple algebraic equations.

For the homogeneous linear equation we have the following theorem:

*A necessary and sufficient condition that the equation*

$$\phi_1(x) = R(x)\,\phi(x)$$

*have a rational solution (other than zero and infinity) is that the congruent zeros and poles of $R(x)$ can be paired, a zero and a pole together. If $R(x)$ is a constant, it must be 1 for the ordinary difference equation or a power of $q$ for the $q$-difference equation.*

That it is a necessary condition is shown as follows: If $R(x)$ has its leftmost pole [zero] at $a$, then either:

(1) $\phi(x)$ has a zero [pole] at $a$ and $\phi_1(x)$ a zero [pole] at $a_{-1}$; to balance this, either $R(x)$ has a zero [pole] or $\phi(x)$ must have a zero [pole] at $a_{-1}$, and $\phi_1(x)$ therefore has a zero [pole] at $a_{-2}$, etc.

Or, (2) $\phi_1(x)$ has a pole [zero] at $a$ and $\phi(x)$ a pole [zero] at $a_1$; then either $\phi_1(x)$ has a pole [zero] at $a_1$ or $R(x)$ has a zero [pole] at $a_1$. If $\phi_1(x)$ has a pole [zero] at $a_1$, then $\phi(x)$ has a pole [zero] at $a_2$, and either $\phi_1(x)$ has a pole [zero] at $a_2$ or $R(x)$ has a zero [pole] at $a_2$, etc.

Thus, to every zero [pole] of $R(x)$ there must be a congruent pole [zero], or $\phi(x)$ will have an infinite number of zeros [poles].

That this is a sufficient condition can be seen from the fact that, if the zeros and poles occur as stated in the theorem, a rational solution can immediately be written out.

By methods similar to those employed in the next section for the study of the non-linear equation, it can be shown that there may be algebraic solutions other than rational solutions of certain linear $q$-difference equations, provided that the expansions of $R(x)$ in the neighborhood of infinity and of the origin start with a constant which is a fractional power of $q$. The method of finding such solutions is given in § 3.

Stridsberg has shown* that if the equation

$$\phi(x+1) = R(x)\,\phi(x) + a, \qquad a = 1 \text{ or } 0,$$

has an algebraically transcendental solution $\phi(x)$, that solution can be written in the form

$$\phi(x) = P(x)\,G(x) + Q(x),$$

where $Q(x)$ is a rational solution of the original equation and $P(x)$ is a rational solution of

$$P(x+1) = \frac{R(x)}{c}P(x), \qquad \lim_{x=\infty} R(x) = c, \ c \neq 0, \ \neq \infty,$$

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* *Loc. cit.*, p. 2, No. 15.

and where $G(x)$ satisfies the functional equation

$$G(x+1) = c\,G(x),$$

and also satisfies a differential equation with constant coefficients.

From the discussion above concerning rational solutions of linear equations, it is easy to find, for any particular equation, whether it has algebraically transcendental solutions.

For the $q$-difference equation Stridsberg has shown that if the equation

$$\phi(qx) = R(x)\,\phi(x) + a, \qquad a = 1 \text{ or } 0,$$

has an algebraically transcendental solution $\phi(x)$, that solution can be written in the form

$$\phi(x) = P(x)\left(G(x) + \mu\frac{\log x}{\log q}\right) + Q(x), \qquad \mu \text{ a constant},$$

where $G(x)$ satisfies a differential equation with constant coefficients and also satisfies the functional equation

$$G(qx) = c\,x^l G(x),$$

where $c$ is a constant and $l$ a rational number. If $\mu \neq 0$, then $c\,x^l = 1$.

Thus, either the equation

$$P(qx) = \frac{R(x)}{c\,x^l}P(x)$$

and the original equation have algebraic solutions or the equations

$$P(qx) = R(x)\,P(x) \quad \text{and} \quad Q(qx) = R(x)\,Q(x) - \mu P(qx) + a$$

have algebraic solutions, according as $\mu$ is or is not zero. Hence, there are algebraically transcendental solutions of the linear $q$-difference equation of the forms considered, only when one of the sets of equations above has algebraic solutions.

From the results of the preceding paragraphs we conclude that the linear ordinary difference equation of the forms considered has only transcendentally transcendental solutions, unless it has a rational solution; and that the linear $q$-difference equation of the forms considered has only transcendentally transcendental solutions, unless one of the sets of equations above has algebraic solutions.

We can now readily see, by use of the theorem concerning rational solutions of the homogeneous equation, that the function $\Gamma(x)$ is transcendentally transcendental, as was proved by Hölder.* Likewise, it is easy to show that the solutions of the equation $\phi(qx) = (1+x)\,\phi(x)$, discussed by Stridsberg, are transcendentally transcendental.

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* *Mathematische Annalen,* Vol. XXVIII (1887), pp. 1–13.

## § 3. *Algebraic Solutions Other than Rational Functions.*

From the equations (M) and (N) it is evident that if there are singularities of the solutions, other than poles, in the finite plane, the origin excepted for the $q$-difference equation, there is an infinite number of such singularities. The only algebraic solutions for the ordinary difference equation are therefore rational functions. The origin may be a branch point in the $q$-difference equation, and its algebraic solutions are therefore rational in $x^{1/j}$, $j \geqq 1$. We have in this section to study only the $q$-difference equation.

In order to find what values $j$ can have, make substitution of a descending power series in $x^{1/j}$ in equation (2), and we have

$$(a_n x^{n/j} + a_{n-1} x^{(n-1)/j} + \dots) (1 + a_n q^{n/j} x^{n/j} + a_{n-1} q^{(n-1)/j} x^{(n-1)/j} + \dots)$$
$$= r_m x^m + r_{m-1} x^{m-1} + \dots\,.$$

Equating coefficients, we have $n/j = m/2$ and

$$a_n^2 q^{n/j} = r_m,$$

for $m > 0$. If $m$ is even, the first exponent is an integer; and if $m$ is odd, it is a multiple of $1/2$. If it is an integer, no fractional exponent can enter the expansion; for, if a fractional exponent $l/j$ enters, the equation for the determination of its coefficient will be

$$a_n a_l (q^{m/2} + q^{l/j}) = 0.$$

If $q$ is not a root of unity, we shall have $a_l = 0$. If $m$ is odd, there can be no other exponents than multiples of $1/2$. Suppose, an exponent not a multiple of $1/2$ enters, say $k/j$. The equation for the determination of the corresponding coefficient will be

$$a_n a_k (q^{n/2} + q^{k/j}) = 0,$$

since $m/2 + k/j$ is not an integer. Hence, if $q$ is not a root of unity, $a_k = 0$.

If $m < 0$, the first exponent will be an integer, and the same argument as that given above will show that no fractional exponent can enter unless $q$ is a root of unity.

If $m = 0$, the first coefficient will be determined by the equation

$$a_0^2 + a_0 = r_0.$$

If a fractional exponent $-l/j$ enters, the equation for determining the corresponding coefficient will be

$$a_{-l} (a_0 + 1 + a_0 q^{-l/j}) = 0.$$

Either $a_{-l} = 0$ or $a_0 = -q^{l/j}/(1 + q^{l/j})$; in the latter case, $r_0 = -q^{l/j}/(1 + q^{l/j})^2$.

Consequently, $j$ is either 1 or 2, except possibly when the expansion for $R(x)$ in the neighborhood of infinity starts with the constant $r_0$ above determined. Similarly, if we used expansions valid in the neighborhood of the origin, we should find that $j$ is either 1 or 2, except possibly when the expansion for $R(x)$ starts with the same constant $r_0$.

We thus have the following theorem:

*If $R(x)$ is infinite of odd order both at zero and at infinity, the equation*

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1,$$

*where $q$ is not a root of unity, can have no algebraic solutions other than those which are rational in $x^{1/2}$ but not in $x$.*

*If the expansions for $R(x)$ in the neighborhood of zero and of infinity start with the constant $-q^{l/j}/(1+q^{l/j})^2$, where $l$ and $j$ are positive integers, there may be solutions rational* in $x^{1/j}$.*

*In all other cases any algebraic solution is rational.*

If $\phi(x)$ is a solution rational in $x^{1/j}$, we can write it in the form

$$\phi(x) = \frac{P(x) + \varepsilon x^{1/j} Q(x) + \varepsilon^2 x^{2/j} S(x) + \ldots\ldots + \varepsilon^{j-1} x^{(j-1)/j} T(x)}{P'(x) + \varepsilon x^{1/j} Q'(x) + \varepsilon^2 x^{2/j} S'(x) + \ldots\ldots + \varepsilon^{j-1} x^{(j-1)/j} T'(x)},$$

where $\varepsilon$ is 1 and $P(x)$, $P'(x)$, $Q(x)$, etc., are rational in $x$. Then the above expression is a solution when $\varepsilon$ is any $j$-th root of unity.

If there are solutions of the normal form (2) which are rational in $x^{1/j}$, then the equation

$$\phi(q^{1/j} x) = \frac{R(x^j)}{\phi(x)} - 1$$

has corresponding solutions rational in $x$. The methods of the section on rational solutions will enable us to determine those solutions. Hence, we can find all the algebraic non-rational solutions of a given equation. A similar process will enable us to find all the algebraic non-rational solutions of the linear equation.

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* That solutions rational in fractional powers of $x$ do exist is shown by the following examples:

For $q=4$ and $R(x) = 2(x - 1/9)$ the normal equation (2) has the solutions

$$\phi(x) = \pm x^{1/2} - 1/3.$$

For $q=8$ and $R(x) = -2/9$ equation (2) has the solutions

$$\phi(x) = -1/3, \quad \phi(x) = -2/3, \quad \phi(x) = -\frac{2x + m x^{2/3} + m^2 x^{1/3} - 1}{3(x-1)}, \quad \phi(x) = \frac{-2(x^{1/3}+a)}{3(x^{1/3}+2a)},$$

where $m$ takes the values of the three cube roots of unity and $a$ is arbitrary.

For $q=8$ and $R(x) = -\frac{2(x^2 - 9x + 1)}{9(x^2 + 9x + 1)}$ equation (2) has the solutions

$$\phi(x) = -\frac{2x + m x^{2/3} + m^2 x^{1/3} - 1}{3(x+1)},$$

where $m$ takes the values of the three cube roots of unity.

Concerning the normal form (3), we have the following theorem:

*If $R(x)$ is infinite of odd order both at zero and at infinity, the equation*

$$\phi(qx) = \frac{R(x)}{\phi(x)},$$

*where $q$ is not a root of unity, can have no other algebraic solutions than those rational in $x^{1/2}$ but not in $x$. In all other cases the only algebraic solutions are rational.*

These algebraic non-rational solutions can be found by finding the rational solutions of

$$\phi(q^{1/2}x) = \frac{R(x^2)}{\phi(x)}.$$

The proof of the above theorem follows the argument for the other normal form so closely that it will not be given here.

## § 4. *Algebraically Transcendental Solutions.*

Stridsberg proved that, if the normal equation (2) has no algebraic solution, its algebraically transcendental solutions, if any exist, also satisfy a Riccati equation with coefficients rational in $x$.

While proving this theorem, Stridsberg seems to have doubted the existence of such functions, as has been noted in the Introduction.

Assuming that $\phi(x)$ satisfies the two equations

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \tag{2}$$

and

$$\phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x) = 0, \tag{8}$$

where $L(x)$, $M(x)$, $N(x)$ are rational, we shall find some conditions which enable us to put the theorem in a different form.

We have from (8), by changing the argument,

$$\phi_1'(x) + L_1(x)\{\phi_1(x)\}^2 + M_1(x)\phi_1(x) + N_1(x) = 0. \tag{9}$$

Substituting in (9) from (2) so as to have only the argument $x$ in the function $\phi$, we have

$$\frac{\phi(x)R'(x) - R(x)\phi'(x)}{\alpha\{\phi(x)\}^2} + L_1(x)\left\{\frac{R(x)}{\phi(x)} - 1\right\}^2 + M_1(x)\left\{\frac{R(x)}{\phi(x)} - 1\right\} + N_1(x) = 0.$$

In this equation and what follows, $\alpha$ is to be considered 1 or $q$, according as we have the ordinary difference or the $q$-difference equation.

Comparing this equation with (8), after clearing of fractions and dividing by the coefficient of $\phi'(x)$, we have the following equations which $L(x)$, $M(x)$, $N(x)$ must satisfy:

$$\left.\begin{aligned} N(x) + \alpha L_1(x) R(x) = 0, \\ M(x) R(x) - 2\alpha L_1(x) R(x) + \alpha M_1(x) R(x) + R'(x) = 0, \\ L(x) R(x) + \alpha L_1(x) - \alpha M_1(x) + \alpha N_1(x) = 0; \end{aligned}\right\} \qquad (10)$$

or, by elimination of $M(x)$ and $N(x)$, we have the equation

$$\begin{aligned} \alpha^3 R_2(x) L_3(x) + \alpha^2 \{R_1(x) + 1\} L_2(x) - \alpha\{R_1(x) + 1\}L_1(x) \\ - R(x) L(x) - \alpha R_1'(x)/R_1(x) = 0. \end{aligned} \qquad (11)$$

This equation puts all the restriction on $R(x)$ that the system of three equations above does. That is, in order that equation (2) have a solution $\phi(x)$ which is also a solution of (8), it is necessary that (11) have a rational solution $L(x)$. This last sentence is simply another statement of the theorem.

We want to prove that this necessary condition for algebraically transcendental solutions is also sufficient. In order to do this, set

$$D(x) = \phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x),$$

where $L(x), M(x), N(x)$ satisfy the relations (10). A necessary and sufficient condition that a solution $\phi(x)$ of the equation (2) also satisfy the Riccati equation, is that $D(x) \equiv 0$. Now,

$$D_1(x) = \phi_1'(x) + L_1(x)\{\phi_1(x)\}^2 + M_1(x)\phi_1(x) + N_1(x).$$

Multiply the equation in $D(x)$ by $R(x)/\{\phi(x)\}^2$, the equation in $D_1(x)$ by $\alpha$, and add, making use of the equations (10). Then we have readily

$$\frac{D_1(x)}{D(x)} = -\frac{R(x)}{\alpha\{\phi(x)\}^2}.$$

This equation shows that, if there is a solution $\phi(x)$ of equation (2), in the case of the ordinary difference equation, which is asymptotic to a power of $x$ times a descending power series in any fractional power of $x$, the only possible value of $D(x)$ is zero, except when the expansion of $R(x)$ in the neighborhood of infinity starts with the constant $-1/4$. In the case of the $q$-difference equation, $q$ not a root of unity, the exception is when the expansions for $R(x)$ in the neighborhood of zero and of infinity each start with the constant $-q/(1+q)^2$.

We shall now determine when equation (2) has solutions which are asymptotic to a power of $x$ times a power series in $1/x$ or in $1/\sqrt{x}$. The substitution

$\phi(x) = \psi_1(x)/\psi(x)$ will transform the normal form (2) to the second-order linear homogeneous equation

$$\psi_2(x) + \psi_1(x) - R(x)\,\psi(x) = 0. \tag{12}$$

We prove in the next section that if $R(x)$ has a zero at infinity, then equation (11) has no rational solution, so that we shall not consider such an $R(x)$ here.

For the ordinary difference equation, if $R(x)$ has a pole of even order at infinity or approaches a constant different from $-1/4$ as $x$ approaches infinity, then the works of Carmichael* and of Birkhoff† show that there is a solution of equation (12) such that, when we form the quotient $\psi_1(x)/\psi(x)$, it has the property of being asymptotic to a power of $x$ times a power series in $1/x$. When $R(x)$ has a pole of odd order at infinity, it can be shown‡ that the linear equation (12) has a solution $\psi(x)$ such that, when we form the quotient $\psi_1(x)/\psi(x)$, it has the property of being asymptotic to a power of $x$ times a power series in $1/\sqrt{x}$.

For the $q$-difference equation, if $R(x)$ has a pole of even order at infinity or approaches a constant different from $-q^t/(1+q^t)^2$, where $t$ is an integer, as $x$ approaches infinity, then the results of a paper by Carmichael§ show that the quotient $\psi_1(x)/\psi(x)$ has the property of being asymptotic to a power of $x$ times a power series in $1/x$. If $R(x)$ has a pole of odd order at infinity, we can replace $R(x)$ by $R(x^2)$ and $q$ by $\sqrt{q}$, and have an equation to which the results of the paper mentioned are applicable. Making the reverse substitution in the solution of the new equation, we shall have for the quotient $\psi_1(x)/\psi(x)$ a power of $x$ times a power series in $1/\sqrt{x}$.

From these considerations it follows that in each case there will be two solutions of equation (12) asymptotic to such formal expansions that, if we form the quotient $\phi(x) = \psi_1(x)/\psi(x)$, it will be asymptotic to a power of $x$ times a power series in $1/x$ or in $1/\sqrt{x}$.

From the results thus obtained we have the theorem:

*If, as $x$ approaches infinity, $R(x)$ does not approach the constant $-\alpha/(1+\alpha)^2$, where $\alpha$ is* 1 *or $q$ according as we have the ordinary difference or the $q$-difference equation, and the equation*

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \tag{2}$$

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* *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 99–134.

† *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 243–284.

‡ Lectures of Dr. Carmichael at Indiana University, 1913–14.

§ AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), pp. 147–168.

*has no algebraic solution, a necessary and sufficient condition that equation* (2) *have a solution which is algebraically transcendental is that the equation*

$$\alpha^3 R_2(x)\, L_3(x) + \alpha^2 \{R_1(x) + 1\} L_2(x) - \alpha \{R_1(x) + 1\} L_1(x) \\ - R(x)\, L(x) - \alpha R_1'(x)/R_1(x) = 0 \qquad (11)$$

*have a rational solution* $L(x)$. *Any solution of equation* (2) *which is asymptotic to one of its formal solutions in powers of* $1/x$ *or of* $1/\sqrt{x}$ *will be algebraically transcendental. It will satisfy the Riccati equation*

$$\phi'(x) + L(x) \{\phi(x)\}^2 + M(x)\, \phi(x) + N(x) = 0,$$

*where* $M(x)$ *and* $N(x)$ *are determined from equations* (10).

For the ordinary difference equation $R(x) = k \dfrac{x(x+3)}{(x+1)(x+2)}$, $k \neq -1/4$, and for the $q$-difference equation $R(x) = k \dfrac{(x+q)(x+q^4)}{(x+q^2)(x+q^3)}$, $k \neq -q^t/(1+q^t)^2$, where $t$ is a rational number, the normal form (2) has no algebraic solution; while equation (11) does have a rational solution. The normal equations, therefore, have as solutions transcendental functions which also satisfy Riccati equations. I expect to develop the theory of these functions in detail.

## § 5. *Equations Having only Transcendentally Transcendental Solutions.*

A result obtained in the preceding section may be restated in the following form:*

*If the equation*

$$\phi_1(x) = \frac{R(x)}{\phi(x)} - 1 \qquad (2)$$

*has no algebraic solution, then all its solutions are transcendentally transcendental, provided that the equation*

$$\alpha^3 R_2(x)\, L_3(x) + \alpha^2 \{R_1(x) + 1\} L_2(x) - \alpha \{R_1(x) + 1\} L_1(x) \\ - R(x)\, L(x) - \alpha R_1'(x)/R_1(x) = 0 \qquad (11)$$

*has no rational solution.*

We shall next consider rational solutions of equation (11). In order to do this, we shall first investigate the relative number of poles and zeros of $L(x)$ by means of series. Make the substitution

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* This theorem is here derived by means of a theorem of Stridsberg quoted in the preceding section. I had, however, proved the theorem in essentially the present form before the work of Stridsberg came to my attention. In view of the results in § 4, it is seen that the two theorems are equivalent.

$$L(x) = l_\lambda x^\lambda + l_{\lambda-1} x^{\lambda-1} + \ldots, \qquad l_\lambda \neq 0,$$
$$R(x) = r_m x^m + r_{m-1} x^{m-1} + \ldots, \qquad r_m \neq 0,$$

in equation (11). Equating the coefficients of the highest powers of $x$ to zero, we shall have for the ordinary difference equation when $m > 0$,

$$r_m l_\lambda (0) - \{m\} = 0,$$

the quantity $\{m\}$ entering only if $m + \lambda = -1$;

$$r_m l_{\lambda-1} (0) + r_{m-1} l_\lambda (0) + r_m l_\lambda (2m + 3\lambda + m + 2\lambda - m - \lambda) - \{m\} = 0,$$

the quantity $\{m\}$ entering only if $m + \lambda = 0$.

It is obvious that these equations can be satisfied only when $\lambda = -m/2$ or $\lambda = -m$.

If $m = 0$, our equations are

$$r_0 l_\lambda (0) + l_\lambda (0) - \{s\} = 0,$$

where $\{s\}$ is a constant $\neq 0$ and enters only if $\lambda \leqq -2$;

$$r_{-1} l_\lambda (0) + r_0 l_{\lambda-1} (0) + l_{\lambda-1} (0) + l_\lambda (2\lambda - \lambda) + r_0 l_\lambda (4\lambda) - \{s\} = 0,$$

where $\{s\}$ enters only if $\lambda \leqq -2$.

These equations can be satisfied by $\lambda = 0$, $\lambda = -k + 1$, where $r_{-k} x^{-k}$ is the second term in the expansion of $R(x)$, or by $r_0 = -1/4$.

If $m < 0$, the equations are

$$l_\lambda (0) + \{m\} = 0,$$

where $\{m\}$ enters only if $\lambda = -1$;

$$l_{\lambda-1} (0) + l_\lambda (2\lambda - \lambda) + \{m\} = 0,$$

where $\{m\}$ enters only if $\lambda = 0$.

These equations can be satisfied for no value of $\lambda$; that is, *if $R(x)$ is zero at infinity,*[*] *there is no rational $L(x)$ satisfying equation* (11).

For the $q$-difference equation the same method gives the following results:

$m > 0$, $\lambda = -m/2 - 1$ or $\lambda = -m - 1$;

$m = 0$, there can exist a rational $L(x)$ satisfying the equation (11) only when $r_0 = -q^{1+\lambda}/(1 + q^{1+\lambda})^2$, where $\lambda$ is an integer;

$m < 0$, *there exists no rational $L(x)$ satisfying the equation.*

Expansions in the neighborhood of the origin will give analogous results for the $q$-difference equation.

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[*] This was the case proved by Tietze.

We shall next consider the possible number and location of the poles of the rational function $L(x)$ satisfying equation (11). If the leftmost zero or pole of $R(x)$ in any congruent set is $a$ and the rightmost is $a_k$, then the leftmost pole of $R_1'(x)/R_1(x)$ in the same set is $a_{-1}$ and the rightmost is $a_{k-1}$. If $L(x)$ has a congruent pole to the left of $a_1$, then $L_3(x)$ has a pole to the left of $a_{-2}$; and since $R_2(x)$ does not have a zero to the left of that point, $L(x)$, $L_1(x)$, or $L_2(x)$ must have a pole at the point at which $L_3(x)$ has a pole. Then $L_3(x)$ has a pole farther to the left, and it must again be balanced and there must be an infinite number of congruent poles. Similarly, one sees that the rightmost pole of $L(x)$ can not be to the right of $a_k$ without introducing an infinite number of congruent poles. Thus, it follows that the congruent poles of a rational solution $L(x)$ in any set can be only at the points $a_1, a_2, \ldots, a_k$, where $a$ and $a_k$ are the leftmost and rightmost congruent zeros or poles of $R(x)$ in that set. Clearly, $L(x)$ can not have a pole which is not congruent to a zero or a pole of $R(x)$. Likewise it can be shown that there can be no rational $L(x)$ for an $R(x)$ which has a zero or pole to which there is no congruent zero or pole, the origin excepted for the $q$-difference equation. The order of any pole can not be greater than the sum of the orders of the congruent zeros and poles of $R(x)$.

We have found an upper bound to the number and order, and have found the possible locations of the poles of $L(x)$, and we know the relative number of poles and zeros, except when the expansion for $R(x)$ starts with the particular constants noted above; we may therefore replace $L(x)$ by $P(x)/Q(x)$, where $Q(x)$ is a fully determined polynomial and $P(x)$ is a polynomial of known degree, with the exceptions noted. We can substitute a polynomial of proper degree for $P(x)$ and determine its coefficients by simple algebraic processes if a rational $L(x)$ exists. In the exceptional cases we can substitute a general polynomial and reckon out its degree as well as its coefficients.

In addition to furnishing methods for determining the character of the solutions of any given equation, the results obtained in this paper enable us to name a number of classes of equations which have only transcendentally transcendental solutions. We shall define the more comprehensive classes in the following theorems:

*The equation*

$$\phi(x+1) = \frac{R(x)}{\phi(x)} - 1$$

*has only transcendentally transcendental solutions, provided that* $R(x)$ *has a* *pole of odd order at infinity and has a zero or a pole to which there is no congruent zero or pole.*

*The equation*

$$\phi(qx) = \frac{R(x)}{\phi(x)} - 1$$

*has only transcendentally transcendental solutions, provided that* $R(x)$ *satisfies any one of the following sets of conditions:*

1) $R(x)$ *has a zero at infinity [zero] and a pole of odd order at zero [infinity].*

2) $R(x)$ *has a pole of odd order at infinity [zero] and a pole of even order at zero [infinity], and has a zero or pole to which there is no congruent zero or pole.*

3) $R(x)$ *has a pole of odd order at infinity [zero], and at zero [infinity] is a constant different from zero, infinity and* $-q^t/(1+q^t)^2$, *where* $t$ *is a rational number.*

We return to a consideration of the normal form (3), $\phi_1(x)\phi(x) = R(x)$. This equation transforms by a linear fractional substitution (see § 1) to the normal form (2); and hence, by § 4, if the normal form (3) has no algebraic solutions, its algebraically transcendental solutions, if any exist, satisfy a Riccati equation. If, as in the preceding section, we assume that a solution $\phi(x)$ of equation (3) also satisfies the Riccati equation

$$\phi'(x) + L(x)\{\phi(x)\}^2 + M(x)\phi(x) + N(x) = 0,$$

we are led in the same way to the following equations which must be satisfied by rational functions $L(x)$, $M(x)$, $N(x)$:

$$\begin{aligned} L(x)R(x) + \alpha N_1(x) &= 0, \\ \alpha M_1(x) + M(x) + R'(x)/R(x) &= 0, \\ \alpha L_1(x)R(x) + N(x) &= 0. \end{aligned}$$

Consider the second of these equations. The poles of $R'(x)/R(x)$ are all of first order, and the residues at those poles are all integers. Consequently, we are led to see that, in order for $M(x)$ to be rational, it must have all its poles of the first order, and the residues at the poles must be integers. These properties of $M(x)$ are sufficient to make the expressions $e^{\int M(x)dx}$ and $e^{\int \alpha M_1(x)dx}$ represent rational functions. If we integrate the equation and write in exponential form, we have

$$e^{\int \alpha M_1(x)dx}\, e^{\int M(x)dx}\, e^{\int R'(x)/R(x)dx} = 1.$$

Write $1/D(x)$ for $e^{\int M(x)dx}$, and the above equation readily reduces to

$$D_1(x)\,D(x)=R(x).$$

Since $D(x)$ is rational, we see that a necessary condition that the equation in $M(x)$ have a rational solution, is that the normal equation (3) have a rational solution. A consideration of the first and third of the set of three equations given above will lead to the same conclusion unless $L(x)$ and $N(x)$ are both zero.

We have then the following theorem:

*If the equation*

$$\phi_1(x)=\frac{R(x)}{\phi(x)}$$

*has no algebraic solution, all its solutions are transcendentally transcendental.*

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